HED summer school lectures on
Dense Plasmas

Outline of lectures

I. Dense Plasma Kinetics

II. Dense Plasma Thermodynamics

III. Energy Transport in Dense Plasma

I. A Introduction and Definition of the plasma parameter in HED plasma

• We are concerned with plasmas of electron density \( n \geq 10^{12} \text{ cm}^{-3} \) (atmospheric density) and electron temperature \( T_e \geq 10 \text{ eV} \rightarrow \sim 1 \text{ keV} \)

• To describe such an HED plasma, we would like to know its fundamental properties

1) Equation of state \( \rho = f(n, T) \) or \( \rho = f(n_i) \)
2) Electrical conductivity \( \sigma(n, T) \)
3) Thermal conductivity \( K(n, T) \)
4) Stopping power for fast particles \( \delta E/\delta x \)
5) Electron-electron and e⁻-ion relaxation times \( \tau_e \)
6) Average charge state, \( Z \)
7) Radiation absorption and emission properties

• In these lectures, we will discuss (1), (2), (3) and (6)
First, we consider the validity of even describing an HED plasma as a plasma.

- Debye shielding in a plasma

\[ \nabla^2 \phi = -\frac{q}{4\pi \rho} \text{ charge density} \]
\[ \nabla^2 \phi = 4\pi e (n_e - n_i) = 4\pi e \phi \delta(x) \]

From stat. mech:

\[ n_e (x) = n_0 e^{\frac{e\phi}{kT_e}} \quad n_i (x) = n_0 e^{-\frac{Ze\phi}{kT_i}} \]

Standard plasma treatment:

Taylor expand \( e^{\frac{e\phi}{kT_e}} \) assuming

\( e\phi \ll kT_e \) i.e. hot and not too dense

\( \Rightarrow \) This assumption fails in a dense plasma

when \( e\phi = kT_e \) the plasma is said to be strongly coupled

\[ \nabla^2 \phi = 4\pi e n_e \left( \frac{e\phi}{kT_e} \right) + 4\pi e n_i \left( \frac{Ze\phi}{kT_i} \right) \]

In spherical coordinates

\[ \frac{1}{v^2} \frac{d}{dv} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi e^2 \phi \left( \frac{n_e}{kT_e} + \frac{Ze n_i}{kT_i} \right) \]
Very close to the change $\psi \sim \frac{1}{r}$

So choose solution such that

$$\psi = \frac{q_i}{r} e^{-r/L} \quad \text{(shielded with length} \ L)$$

Substituting

$$\psi = \frac{q_i}{r} e^{-r/L} - \frac{q_i}{L} \frac{e^{-r/L}}{r} + \frac{q_i}{L} \frac{e^{-r/L}}{r^2} = 4\pi e^2 \frac{Z_n}{L} \left( \frac{1}{kT_e} + \frac{1}{kT_i} \right)$$

This leads to

$$\lambda_D = \sqrt{\frac{kT_e + kT_i}{4\pi e^2 Z_n}}$$

Debye shielding length

Traditional plasma formulation only works when there are many particles in a Debye sphere

So we define the plasma parameter

$$\Lambda_p = N_e \lambda_D^3 = \# \, \text{of electrons in a cube} \ \lambda_D^3$$

ignoring $T_e$ (i.e. 20)

$$\Lambda_p = \frac{1}{\sqrt{Z_n}} \left( \frac{kT_e}{4\pi e^2} \right)^{3/2}$$

Denser plasmas have smaller $\Lambda_p$

Consider some examples. Note that $\lambda_D = 740 \sqrt{\frac{kT_e (eV)}{n_e (cm^{-3})}}$

1) MFE plasma $\Lambda_p = 4 \times 10^8 \left[ \frac{kT_e (eV)}{n_e (cm^{-3})} \right]^{1/2} \text{cm}$

$kT_e = 10,000 \text{ eV} \quad n_e = 10^{22} \text{ cm}^{-3} \implies \Lambda_p = 4 \times 10^7$

2) HFE plasma $kT_e = 100 \text{ eV} \quad n_e = 10^{22} \text{ cm}^{-3}$

$$\Lambda_p = 4 \quad \text{!! Not really a transition plasma}$$
I. § Coulomb collision frequency

- Dynamics in dense plasmas are dominated by collisional processes.

→ We will skip the full kinetic theory to derive a useful formula quickly.

• Our goal is to find the frequency at which electrons undergo collisions with ions.

Consider this scattering geometry:

\[ e^- \rightarrow p \]

- How much perpendicular velocity is acquired?

\[ mV_p = \int_{-\infty}^{\infty} F_\perp(t) \, dt = \text{total impulse} \]

• We need to differentiate between "small" and "large" angle collisions.

Small angle collision: \( V_\perp \ll V_0 \), small deviation angle.

This will occur if \( KE \gg \text{potential energy} \)

\[ \frac{1}{2} m V_0^2 \gg Z e^2 / p \]

at \( p_0 \). These are equal

\[ p_0 = \frac{2 Z e^2}{m V_0^2} \quad \text{Larmor length} \]

when \( p \approx p_0 \) → small angle coll.

\( p < p_0 \) → large angle.

In "standard" plasma theory small angle collisions dominate.
So we consider the effects of many small angle collisions. 
This is only a fair assumption in HED plasmas.

But we consider \( p > p_0 \).

In this case we can evaluate the integral as if it were a straight line path:

\[
F_\perp = \frac{Ze^2}{r^2} \sin \theta \quad \text{with} \quad \theta = \frac{p}{p_0}
\]

\[
F_\parallel = \frac{Ze^2}{p^2} \sin \theta
\]

\[
V_\perp = \int_0^\theta \frac{Ze^2}{mc^2p_0^2} \sin \theta \, d \theta = \frac{Ze^2}{mc^2p_0^2} \theta
\]

Since \( t = \frac{d \theta}{\sin \theta} \),

\[
z = -v_0 \cos \theta = -v_0 \frac{p}{p_0} \sin \theta
\]

\[
\frac{dt}{dt} = \frac{p}{V_0 \sin \theta}
\]

\[
V_\perp = \frac{Ze^2}{mc^2p_0^2} \int_0^\theta \sin \theta d \theta = \frac{Ze^2}{mc^2p_0^2} \theta
\]

Effect of a single collision:

Now many collisions:

Many small kicks, \( \Delta x \): yield

\[
\Delta x_{\text{tot}} = \sum_{i=1}^{N} \Delta x_i
\]

\[
\langle \Delta x_{\text{tot}}^2 \rangle = \langle \left( \sum_{i=1}^{N} \Delta x_i \right)^2 \rangle
\]

Since kicks are uncorrelated \( \langle \Delta x_i \cdot \Delta x_j \rangle = 0 \)

\[
\langle \Delta x^2 \rangle = N \langle \Delta x_i^2 \rangle
\]

Now use the fact that \( \frac{V_\perp}{V_0} = \frac{p}{p_0} \).
\[ \langle V_+^2 \rangle = \frac{V_0^2 \rho_0^2}{\rho^2} = \langle \Delta V_+^2 \rangle + \langle \Delta V_-^2 \rangle \]
\[
\langle V_{+tot}^2 \rangle = N V_0^2 \rho_0^2 / \rho^2
\]

We want a rate so we need a time derivative

\[ \frac{d}{dt} \left( \frac{2 \pi p dp}{\text{impact area}} \right) = \frac{2 \pi p dp}{\text{impact area}} n_0 V_0 \]

so

\[ \frac{d}{dt} \langle V_{+tot}^2 \rangle = \frac{V_0^2 \rho_0^2}{\rho^2} \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} dp \]

To get the total rate of change of \( \langle V_+^2 \rangle \), we need to integrate over all possible impact parameters

\[ \frac{d}{dt} \langle V_{+tot}^2 \rangle = 2 \pi n_0 V_0^3 \rho_0^2 \ln \left( \frac{\lambda_0 / \rho_0}{2} \right) \]

Coulomb logarithm = \( \Lambda_c \)

Note that if \( V_0 \approx \sqrt{kT_e/e m_e} \)

\[ \Lambda_c = \frac{(kT_e)^{1/2}}{2Ze^2 \sqrt{4\pi e^2 n_0}} \approx \frac{2 \pi \Lambda_p}{Z} \text{ (plasma parameter)} \]

MFE plasma \( \ln \Lambda_c \approx 20 \)

HED plasma \( Z=1 \) \( n_e = 10^{22} \text{ cm}^{-3} \), \( kT_e = 100 \text{ eV} \), \( \ln \Lambda_c \approx 3 \)

\( Z=10 \) \( n_e = 10^{23} \text{ cm}^{-3} \), \( kT_e = 10 \text{ eV} \), \( \ln \Lambda_c = -4 \)

Theory obviously breaks down at high density.
So, to find the collision frequency $\nu_c$:

$$\frac{1}{t} \langle \nu_{ei} \rangle = \nu_c \nu_0^2 = 2\pi n_0 \nu_0^3 \left( \frac{Z^2 e^4}{m_e v_0^2} \right) \ln \Lambda_c$$

$$\nu_c = \frac{6\pi}{3^{3/2}} \frac{Z^2 e^4 n_i}{m_e v_0^2} \ln \Lambda_c$$

* This can be thought of as a momentum relaxation frequency.

If we say that $\nu_c = \frac{1}{2} \frac{m_e v_0^2}{\sqrt{2kT_c}}$ the average velocity $\nu_0 = \sqrt{3kT_c/m_0}$

$$\nu_c = \frac{6\pi}{3^{3/2}} \frac{Z^2 e^4 n_i}{m_e v_0^2} \ln \Lambda_c$$

This is very close to relaxation times found by more sophisticated kinetic theory, i.e., our result $\nu_c \approx 4.6 \frac{Z^2 e^4 n_i}{m_e v_0^2} \ln \Lambda_c$ within 40%.

* Note $\nu_c \sim \frac{1}{(kT_c)^{3/2}}$ which means relaxation time $t_e \sim (kT_c)^{-3/2}$

** hotter plasmas are less collisional.

Finally note

$$\frac{\nu_c}{\nu_{pe}} = \frac{6\pi}{3^{3/2}} \frac{Z^2 e^4 n_i}{m_e v_0^2} \ln \Lambda_c \frac{\sqrt{m_0}}{4\pi e^4 n_p}$$

$$= 4\pi^{1/2} Z \frac{1}{\Lambda \rho}$$

MFE plasmas $\nu_c/\nu_{pe} \ll 1$ HE plasmas $\nu_c/\nu_{pe} \sim 1$
I. C Plasma conductivity

Let's apply this concept to a conductivity theory.

**Ohm's law:** \[ \bar{E} = \rho \bar{J} \]

\[ \bar{E} = \frac{5}{10^5} \text{ conductivity} \]

\[ e^+ \rightarrow e^- \text{ is accelerated by the E-field for a time between collisions} \]

\[ n_e m_e \frac{\bar{v}_e''}{v_e} = n_e e \bar{E} \text{ Newton's law} \]

\[ \bar{v} \approx 1/\nu \text{ electron momentum relaxation time via e- ion collisions} \]

\[ \bar{J} = n_e e \bar{v}_e'' \]

\[ \bar{E} = \frac{m_e}{e^2 \nu_e} \bar{J} \]

\[ \bar{J} \rightarrow 10 \rightarrow \sigma = \frac{e^2 m_e}{2} \frac{1}{\nu_e} \]

So the conductivity is:

\[ \sigma = \frac{1}{9} \frac{n_e}{(k T_e)^{3/2}} \frac{1}{2 \pi e^2 m_e^{1/2} \ln \Lambda_e} \text{ Cgs units of sec}^{-1} \]

Independent of density

- higher temperature is more conductive

Consider solid density Cu

- conductivity @ room temp = 5.6 \times 10^{-9} (\Omega m)^{-1}

At what Te is Cu plasma conductive?

Assume it is stripped to the L shell

\[ Z = 19 \]

\[ k T_e = 8 \text{ keV} \Rightarrow \text{ so a plasma is not very conductive} \]
\[ \sigma \text{ at solid density} \]

\[ \ln \sigma \]

\[ \frac{1}{T_c} \]

\[ \sim (kT_c)^{3/2} \]

\[ \ln (kT_c) \]

1) **Coulomb logarithm**

\[ \ln \Lambda_c \sim \ln \left( \frac{(kT_c)^{3/2}}{m^2} \right) \]

Goes down as \( kT_c \) goes down and \( \Lambda_c \) gets dense

\[ \sigma \sim \frac{1}{\ln \Lambda_c} \]

\[ \ln \Lambda_c \]

\[ \ln kT_c \]

\[ \ln \sigma \]

2) **Atomic physics and ionization**

\[ \sigma \sim \frac{1}{Z} \]

Higher average \( Z \) decreased \( e^- \) mean free path

\[ Z \text{ varies with temperature } Z(T_c) \]

\[ Z \]

\[ Z \text{ closed shells} \]

\[ T_c \]

\[ \ln \sigma \]

\[ kT_c \]

Full in \( T_c \)

\[ \text{leads to fill in } kT_c \]

\[ \text{increases conductivity} \]
3) **Electron degeneracy effects**

At high \( n_e \to \) degeneracy is important

\[
\frac{k T_F}{E} \quad \text{only the higher \( E \) electrons participate}
\]

\[
\bar{\epsilon} \approx 1 \text{eV at solid density}
\]

\( T_F \) is Fermi temperature.

4) **Break down of two body collision assumption**

As \( T_e \to 0 \) we begin to overestimate the collision frequency

\[
\frac{\epsilon}{k T_F} \quad \text{MFP is actually larger than ion spacing}
\]

\[ V_{\text{crit}} = \frac{V_e}{R_0} \quad \text{ion mean spacing} \]

\[ R_0 = \left( \frac{4}{3} \pi n_i \right)^{-1/3} \]

\( V_e \) is a thermal velocity

\[
V_e \approx \sqrt{\frac{2}{m_e}} \left[ \frac{(k T_F)^2}{(3/2) k T_e^2} \right]^{1/2}
\]

\( V_e \approx \sqrt{3 k T_e / m_e} \)

we say that \( \gamma_{\text{crit}} = V_{\text{crit}} \)

we set \( \gamma = \frac{V_e}{R_0} \)
So below this critical point

\[ \sigma_{\text{cold}} = \frac{e^2 n e}{mc} \left( \frac{1}{3\pi n_e} \right)^{1/3} \left( \frac{3kT_e}{m_e} \right)^{1/2} \]

so then \( \sigma \sim \frac{2}{3} \frac{n_e^{2/3}}{(kT_e)^{1/2}} \)

\[ \ln kT_e \]

At what temp does this change over for solid Al

\[ \sqrt{34kT_e} \]

\[ n_e = 6 \times 10^{25} \text{ cm}^{-3} \]

\[ Z = 10 \]

\[ \ln \Lambda_e = 1 \]

\[ kT_e \approx 120 \text{ eV} \sim 10 - kT_F \]

Note that this dense regime is one of strong coupling.

\[ E_T = \text{thermal energy} = kT \]

\[ E_p = \text{average particle potential energy} = Z e^2 / \Lambda_0 \]

in "standard" plasma \( E_T \gg E_p \)

So we can define the strong coupling parameter...
\[ \Gamma = \frac{E_p}{E_T} \]

\[ \Gamma^* = \frac{Z^2 e^2}{(\frac{1}{2} \pi n \hbar)^{1/3} kT} \]

when \( \Gamma \ll 1 \) thermal energy dominates
when \( \Gamma \gg 1 \) potential energy dominates
suggested to be strongly coupled

\[ \sqrt{\text{Correlations}} \]

liquid usually has \( \Gamma \approx 100 \)

Al phase diagram

\[ 10^6 \text{ keV} \]
\[ 1 \text{ keV} \]
\[ 0.1 \text{ keV} \]
\[ 0.01 \text{ keV} \]
\[ 0.001 \text{ keV} \]
\[ 1 \text{ eV} \]
\[ 10^{-4} \text{ solid} 100x \]
II. Dense Plasma Thermodynamics

II. A. Dense plasma equation of state

The equation of state requires us to calculate pressure and internal energy:

\[ P(n, T) = P_i + P_e + P_{rad} \]

\[ E(n, T) = E_i + E_e + E_{rad} = \text{internal energy per unit volume} \]

Contributions from electrons, ions, radiation

1) Ions

Ions usually represent a small contribution to the EOS in most HED plasmas. If \( \Gamma < 1 \), we can use an ideal gas EOS:

\[ P_i = n_i k T_i \quad E_i = \frac{3}{2} n_i k T_i \]

2) Electrons

For plasmas not too hot, the electrons determine the EOS.

First, we will consider an EOS in a regime in which the ions are fully ionized or only ionize under changes of temp and pressure.

2.a) Low density

We can use a polytropic equation of state:

\[ P_e = n_e k T_e = Z n_i k T_i \]

\[ E_e = \frac{P}{\gamma - 1} \quad \gamma = \text{ratio of specific heats} \]
Recall from statistical mechanics that
\[ Y = 1 + 2/n \]
\[ n = \# \text{ of degrees of freedom} \]

in the absence of ionization, \( n = 3 \)
\[ Y = \frac{5}{3} \]
\[ E_c = \frac{3}{2} n_e k T_e \]

But this is not correct if the ions are ionized.

Energy must flow into the ionization process.

Then
\[ E_c = \frac{3}{2} n_e k T_e + \sum_{i} \left( n_i \cdot \frac{1}{2} k T_i \right) \]

\[ E_c = \frac{3}{2} n_e k T_e + \sum_{i} \left( n_i \cdot \frac{1}{2} k T_i \right) \]

So in a high-Z plasma, \( E_c > \frac{3}{2} n_e k T_e \).

So if we think in terms of an effective adiabatic index, then
\[ Y < \frac{5}{3} \text{ under ionization conditions} \]

2) Z, b.) High Density \( \Rightarrow \) Degeneracy Effects

At high density, quantum effects are important.

\[ \Rightarrow \text{The electrons fill energy levels by the Pauli exclusion principle} \]

When \( T_e \) is low, the electrons fill levels up to the Fermi energy.

\[ E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N_e}{2} \right)^{2/3} \]

In practical units
\[ E_F = 7.9 \times 10^{-3} \left( \frac{n_e}{1 \times 10^{13} \text{ cm}^{-3}} \right)^{2/3} \]

So at solid \( \text{LET} \) \( E_F \approx 6 \text{ eV} \)
The electron energy distribution function, \( f(E) \), looks like this:

![Graph showing the electron energy distribution function with lines for \( kT_c = E_F \) and \( kT_c = 10E_F \).]

\[ \frac{E}{E_F} \]

The electron pressure still follows a polytropic EOS:

\[ P_e = (\gamma - 1) n_e E_{\text{int}} \]

where \( \gamma = 5/3 \)

So, the internal energy at a fully degenerate electron plasma will be when \( kT_c \ll E_F \)

Uniform momentum distribution:

\[ E_{\text{int}} = \frac{1}{3} \int_0^{P_F} \frac{p^2}{2m_e} \cdot 4\pi p^2 \, dp \]

\[ = \frac{2\pi^2}{5m_e} P_F^5 \left/ \frac{4}{15} \pi P_F^3 \right. \]

\[ E_{\text{int}} = \frac{3}{5} E_F \]

Fermi degenerate pressure:

\[ P_e = \frac{(3\pi^2)^{2/3}}{2} \frac{\hbar^2}{m_e} \frac{n_e^{5/3}}{5} \]

So:

\[ P_e = \frac{2}{3} n_e E_{\text{int}} = \frac{2}{5} n_e E_F \]
Essentially, \( P_e \) is as low as the pressure can be in an electron plasma.

In ICF, one wishes to compress plasma isentropically at as low a pressure as possible, so \( T_e \) must be kept below \( E_F \) by minimizing entropy input.

\[
\frac{P_e}{P_F} = \frac{5}{2} \left( \frac{kT_e}{E_F} \right) + \frac{0.27}{1 + 0.27 \left( \frac{kT_e}{E_F} \right)^{1.044}} \left( \frac{kT_e}{E_F} \right)^{0.27}
\]

A fit given by Atzeni and Meyer-ter-Vehn can be calculated numerically with a Fermi-Dirac distribution.

\[
\frac{P_e}{P_F} \sim kT_e
\]
3) Radiation

At very high temperatures in a dense plasma, if it is large enough, that it is optically thick, it looks like a black body, radiation dominates the pressure.

\[ T_a = \text{radiation temperature} \]

Pressure is from the "photon gas"; it has six degrees of freedom: three for momentum and two for polarization.

\[ Y = \frac{4}{3} \]

\[ E_{\text{rad}} = 3 \, R_{\text{tot}} \]

\[ P_{\text{rad}} = \frac{\pi^2}{k^3 c^3} \left( k T_a \right)^4 = \frac{4}{3} \frac{\pi^2}{k^3 c^3} \frac{\nu_0^4}{k} T_a^4 \]

We can ask at what temperature does \( P_{\text{rad}} = P_e \)

\[ \frac{\pi^2}{k^3 c^3} \left( k T_a \right)^4 = n_e \left( k T_0 \right)^4 \]

Let's assume \( T_a = T_e \); i.e., radiation is strongly coupled to material.

\[ (kT)_{\text{rad-elec}} = \frac{k T_a}{\pi^2 c^3} \]

At solid density, \( n_e = 10^{29} \text{ cm}^{-3} \)

\[ (kT)_{\text{elec}} = 430 \text{ eV} \]

But note: opacity is important so size of system counts.

\[ x_e >> X_{\text{rad}} \ll \Delta Z \]

\[ \text{mean free path} \]

So usually more important in Astrophysics.
We now need to consider the situation in which real, high Z, atoms are in the dense plasma.

We would like to know how ionized the medium is:

\[ \frac{Z}{n} \approx \frac{Z}{n} \approx n \times kT_e \]

\[ \rho \approx \frac{1}{n} \]

First, a few words on local thermodynamic equilibrium (LTE)

LTE = electrons populate quantum levels by Boltzmann statistics

Global equilibrium may not exist

\[ \frac{n_e}{n} \]

If collisions predominate around a particular ion \( \rightarrow \) LTE

Aside: We will treat high Z ions like H-like ions for simplicity

\[ E_n = \frac{Z^2}{n^2} I_n \leq 13.6 \text{ eV} \]

\[ \frac{I_p(z)}{n} \approx \frac{Z^2}{n^2} \]

Principal quantum number

For LTE we demand that collision rates of transitions greatly exceed radiative rates

1. Radiative decay rate = Einstein A coefficient

Integrate over all lower levels \( R_{\text{rad}} = \sum n_m A_m \)

\[ R_{\text{rad}} \approx 1.6 \times 10^{10} Z^4 / n^{3/2} \text{ sec}^{-1} \]

see Greiner
Collisional excitation rate
complex... but

\[ \sigma_{ne} = \frac{2\pi^3}{m_e} \frac{\hbar^4}{d_0^4} \frac{1}{v^2} \frac{F_{\text{osc}}}{\Delta \epsilon_{\text{osc}}} \]

electron velocity
osc. strength

This leads to the estimate that

\[ R_e \approx 3.2 \times 10^{-6} \frac{n_e e^2}{Z^2 \sqrt{k T_e}} \]

in eV

For LTE, \( R_e > R_{\text{rad}} \)

\[ n_e \text{[cm}^{-3}] \geq 5 \times 10^{15} Z^6 \sqrt{k T_e \text{[eV]}} n^{-3/2} \]

\[ \text{fully stripped } N_{\alpha} \]

\[ \text{MUCH easier to achieve LTE in high lying levels} \]

An example: \( Z = 10 \), \( k T_e = 100 \text{ eV} \)

Full LTE down to the ground state, \( n = 2 \)

\[ n_e \approx 2 \times 10^{21} \text{ cm}^{-3} \] - few % of solid density

\( \Rightarrow \) lower electron densities demand a collisional radiative model

\[ \Rightarrow \text{we now consider LTE} \]

\[ \text{Saha equilibrium} \]

\[ \Rightarrow \text{we would like to find the distribution of charge states in a plasma of a given atomic density and temperature} \]

\[ \text{ion density} \rightarrow \Rightarrow \text{electro plasma equations} \]

\( k T_e \leftrightarrow n_e \leftrightarrow n: \text{will be coupled} \)
In LTE we can use statistical mechanics.

We will assume that fields in the plasma do NOT affect quantum levels.

From statistical mechanics:

If \( \bar{N}_n \) is density in the \( n^{th} \) state

\( \bar{N} \) is total ion density

\( E_n \) is the energy of the \( n^{th} \) level

\( W(T) \) is the partition function

\[
\bar{N}_n = \frac{y_n \exp\left[\frac{-E_n}{kT}\right]}{W(T)}
\]

\( y_n \) = degeneracy

\( E_n \) = energy above ground state

on between two distinct state densities

\[
\frac{\bar{N}_n}{\bar{N}_m} = \frac{y_n}{y_m} \exp\left[\frac{-\left(E_n - E_m\right)}{kT}\right]
\]

What we really want is the ratio of free electrons in the continuum (i.e., the plasma density) to the density of a particular charge state.

\[
\frac{dN_e}{N_{n-1}} = \frac{y_e}{y_n} \exp\left[\frac{-\left(E_{n-1} + E_e\right)}{kT}\right]
\]

\( y_e \) = the number of free electron states in some energy interval \( \Delta \epsilon \)

\( E_{n-1} \) = binding energy of \( n \)-th level

### Detailed derivation but

\( K_e = \frac{m_e v_e}{\hbar} \)

\( E_h = \frac{\hbar^2 k^2}{2m_e} \)

\# of modes = \[ \frac{Z \hbar^2}{2k} \frac{dk}{\hbar} \int \left( \frac{2\pi}{L} \right)^3 \]
Wave hands and say that

\[ N_i^2 = \frac{1}{L^2} \]

\[ \text{sensibily} \]

\[ \Psi^2 = \frac{g_i^2 \omega_{ci}^2 \frac{E_p}{2 \pi^2 \hbar^2 N_i^2}}{\frac{2 \pi^2 \hbar^2}{2 \pi^2 \hbar^2 \kappa T} \exp \left( -\frac{E_p}{2 \pi^2 \kappa T} \right)} \]

Plug in, integrate over \( dE_k \) to retrieve \( N_e \)

\[ \frac{N_e N_i^2}{N_i^{2i}} = \frac{2}{2 \pi^2 \hbar^2} \left( \frac{m_e \kappa T_e}{2 \pi^2 \hbar^2} \right)^{3/2} \exp \left[ \frac{-E_p}{2 \pi^2 \kappa T_e} \right] \]

This is a useful form to find the population of electrons in state \( n \) of \( Z = \frac{1}{2} \) ion with an electron 

ion of \( N_i^2 \) and initial ion \( N_i^{2i} \)

\[ \to \quad \text{we would like an eq. that relates total densities of charge state } Z \]

\[ \text{recall that } \quad \frac{N_i^2}{N_i^{2i}} = \frac{\Psi_i^2}{\Psi_i^{2i}(T)} \exp \left( -\frac{E_i^{2i}}{2 \pi^2 \kappa T_e} \right) \]

\[ \frac{N_e N_i^2}{N_i^{2i}} = 2 \frac{W_2(T)}{W_2^{2i}(T)} \left( \frac{m_e \kappa T_e}{2 \pi^2 \hbar^2} \right)^{3/2} \exp \left[ -\frac{E_i^{2i}}{2 \pi^2 \kappa T_e} \right] \]

\[ \exp \left[ -\frac{E_p}{2 \pi^2 \kappa T_e} \right] \]

This is the \( \text{Saha eq.} \)

They yield a coupled set of equations for all charge state densities

\[ N_0, N_1, N_2, N_3 \text{ etc} \]

\[ \text{plus quasi neutrality } N_e = N_1 + N_2 + N_3 \text{ etc} \]

These must be solved numerically
They yield a charge state distribution

\[ N(z) \]

\[ Z \quad \text{Fully} \quad \text{Stripped} \quad \text{to be ionized} \]

For many applications we just want the average ionization state

\[ \bar{Z} = \frac{Z N}{\sum Z N} \Rightarrow N_e = \bar{Z} N_0 \]

A simple approximation can be found for very high Z/nec plasmas (say Au or U):

1) Approximate \( \bar{Z}_{p} = Z \bar{Z}_{H} \) i.e. treat all charges as H-like.

2) Assume that there are two fictitious ions: with \( Z = \bar{Z} + \frac{1}{2} \) and \( Z = \bar{Z} - \frac{1}{2} \)

\[ N(z) \]

\[ N^{Z+1/2} \sim N^{Z-1/2} \]

Therefore

\[ \frac{N_{z+1/2}}{N_{z-1/2}} = 1 \]

So from Sum of these fictitious ions

\[ N_e \cdot 1 = \sum Z \frac{W_{Z+1/2}}{W_{Z-1/2}} \left( \frac{m_e k T_e}{2 \pi \hbar^2} \right)^{3/2} \exp \left[ -\frac{(Z)^2 I_{10}}{k T_e} \right] \]

say that \( W_{Z+1/2} \approx W_{Z-1/2} \) and solve for \( \bar{Z} \)
The in sphere radius is $a = \frac{1}{\sqrt{2}}$. Let's consider a dense plasma in the ion sphere model. This is small and is a good assumption for dense plasma. We can think of a high $Z$ plasma in the ion sphere model. Let's consider the following:

$$\frac{1}{Z} \ln \left[ \frac{Z}{n_e Z} \right]$$

Numerically this is:

$$\frac{1}{Z} \ln \left( \frac{Z}{n_e Z} \right) \approx 0.54 \frac{kT_e}{k}$$

where $kT_e$ is in eV and $n_e$ is cm$^{-3}$. We have $kT_e = 1000$ eV and $n_e = 10^{20}$ cm$^{-3}$. This is very different from the previous case.
How do Coulomb effects change the pressure
recall thermo dynamics
\[ F = -kT_e \ln \frac{W(T_e)}{n_e} - F_{\text{root}} \]

\( F_{\text{root}} \) has two contributions

\[ F_{\text{true}} = -\frac{3}{2} \frac{Z^2 e^2}{Z R_0} \]
\[ F_{\text{rep}} = +\frac{3}{5} \frac{Z^2 e^2}{R_0} \]

\[ F_{\text{root}} = -\frac{9}{16} \frac{Z^2 e^2}{R_0} \]
\[ P_{\text{root}} = -\frac{1}{V} \frac{\partial}{\partial V} F_{\text{root}} = \frac{2}{V} \left( \frac{9}{10} \frac{Z^2 e^2}{R_0} \right) = -\frac{3}{10} \frac{Z^2 e^2}{V^{4/3}} \]

So actual pressure of a dense plasma, at high \( Z \)
\[ \mu = Z n_e kT_e - \frac{3}{10} Z^2 e^2 n_e \frac{V^{4/3}}{R_0} \]

\( \Rightarrow 2) \) Ionization energy effects

We can estimate the extent that energy going into ionization has

\( \text{Recall} \ E_{\text{ion}} = \frac{1}{2} n_e kT_e + \sum_j n_j \left[ \frac{j^2}{2} \bar{I}_j(n) \right] \]

\[ E_{\text{KE}} - E_{\text{ion}} \]

We can assume crudely that all ions are \( A \)-like and that the plasma is ionized all charge states completely up to \( Z \)
\[ E_{\text{ion}} \approx n_e \sum_j \frac{j^2}{2} I_H = \frac{I_H}{6} Z (1+Z) (1+2Z) \]
\[ \approx \frac{1}{6} \frac{1}{Z} Z (1+Z) (1+2Z) \approx \]
\[ \approx \frac{1}{3} Z^3 I_H n_e \text{ for high } Z \]
recalling that \[ \bar{z} = \alpha \sqrt{\frac{kT_e}{I_n}} \] where \( \alpha = 2 \)

\[ E_{en} \approx \frac{3}{2} n_e kT_e \left( \frac{kT_e}{I_n} \right)^{\frac{3}{2}} \quad \frac{3}{2} \alpha^3 \approx \frac{3}{2} \approx 3 \]

so ionization energy is roughly equal to the kinetic energy

**Final note:** This ionization energy affects the effective polytropic index

\[ E_{en} = \frac{2}{\gamma - 1} \]

\[ (\frac{3}{2} + \frac{3}{2} \alpha^2) n_e kT_e = \frac{n_e kT_e}{\gamma_{eff} - 1} \]

\[ \gamma_{eff} \approx 1 + \frac{1}{\frac{3}{2} + \frac{3}{2} \alpha^2} \approx 1.3 \]

So in a high Z, ionizing plasma \( \gamma \) drops from 5/3 down to nearly 1.3

**II. E Atomic physics in a dense plasma**

The fields from particles in a dense plasma affect atomic structure

1) Line broadening
   - electron collisions
   - ion field

2) Continuum lowering
   - \( I_0 \) becomes less
We can approximate this by calculating the attractive energy associated with electrons around the ion.

Two approaches:

1) Debye shielding
2) Ion sphere model

(a) Does not work well in HED plasmas
\[ n_p \sim 1 \]

For sphere model:

\[ R_0 = \left( \frac{3}{4 \pi n_e} \right)^{1/3} \]

So attractive energy between \( e^- \) and ion:

\[ E_{at} = \int_0^{R_0} \frac{Z e^{-} e^+ \cdot 4 \pi r^2 \, dr}{4 \pi \varepsilon_0} \]

\[ = - \left( \frac{Z e^2}{R_0} \right) \int_0^{R_0} \frac{Z e^2}{r^2} \, dr \]

\[ = - \frac{Z e^2}{2} \frac{e^2}{R_0} \]

The repulsive energy between electrons is:

\[ E_{rep} = \int_0^{R_0} \frac{Z e^{-} e^+ \cdot 4 \pi r^2 \, dr}{4 \pi \varepsilon_0} \]

\[ = \frac{Z e^2}{2} \frac{e^2}{R_0} \]

\[ E_{Coul} (\frac{Z^2}{10} \frac{e^2}{R_0}) = \text{Coulomb energy associated with ion in the plasma} \]
We can then estimate the change in ionization potential as the difference between the Coulomb energies of the \( Z \) and \( Z-1 \) ions:

\[
\Delta I_p^{(\infty)} = E_{\infty, Z} - E_{\infty, Z-1}
\]

\[
= -\frac{9}{10} \left( \frac{(Z-1)e^2}{\hbar} \right) - \left( \frac{9}{10} \frac{Ze^2}{\hbar} \right)
\]

\[
= \frac{9}{10} \frac{(Z-1)e^2}{\hbar} \approx \frac{9}{5} \frac{Ze^2}{\hbar}
\]

\[
\Delta I_p^{(\infty)} \approx \frac{9}{5} \left( \frac{4\pi}{3} \right)^{1/3} \frac{Ze^2 n_i^{1/3}}{n_i^{1/3} \rightarrow 3Ze^2 n_i^{1/3}}
\]

Note at high pressure \( \Delta I_p^{(\infty)} \sim I_p^{(\infty)} \).

In this case, the ion is ionized by continuum lowering. This is called pressure ionization.

When is this important?

In a cold plasma \( \Rightarrow \Delta I_p \sim I_p \)

\( 3Ze^2n_i^{1/3} = Ze^2I_p \)

A high-Z plasma will be ionized to \( Z \approx \frac{Ze^2n_i^{1/3}}{I_p} \)

\( Z \approx 1.5 \text{ at } n_i = 10^{22} \text{ cm}^{-3} \)

The density at which pressure ionization is important in a high-Z plasma is

\[
n_i = \frac{1 - \frac{\alpha^2}{27} \eta_i^{3/2} (kT_e)^{3/2}}{27 e_b}
\]

At 100 eV this density is \( \sim 5 \times 10^{24} \text{ cm}^{-3} \).
III. Energy Transport in Dense Plasma

Consider a dense plasma slab

\[ \nabla T_e \quad \text{heating radiation} \]

There will be a flow of thermal energy to the left

Energy will be conducted by
1) electrons - electron thermal transport
2) photons - radiation transport

Energy balance demands:
\[ n_e \frac{\partial T_e}{\partial t} = - \nabla \cdot \bar{S} \]

If \( \nabla T_e \) is not large, it is common to take
\[ \bar{S} = - \chi \nabla T \Rightarrow \frac{\partial T}{\partial t} = \frac{\chi}{n_e} \nabla^2 T \quad \text{steady flow} \]

\( \chi \) depends on \( T \)

III A. Electron heat conduction - Simple model

Mean energy across the plane
from left = \( \frac{1}{6} n_e \text{Ve} E(z-) \)
from right = \( \frac{1}{6} n_e \text{Ve} E(z+) \)
\[
S = \frac{1}{2} ne v_e \left[ E(z-x) - E(z+x) \right] \\
= -\frac{1}{2} ne v_e \lambda \frac{2E}{2z} \\
= -\frac{1}{2} ne v_e \lambda \frac{2E}{2T} \frac{2T}{2z} \\
\Rightarrow S = -\frac{1}{2} \nabla T_e \quad \Rightarrow \quad S = \frac{1}{3} ne v_e \lambda c_v \\
\text{we need an electron MFP } \lambda \\
\text{we can use our estimate for Coulomb scattering time} \\
\lambda_e = \frac{V_e}{V_e} = \frac{1}{2} m_e v_e^2 = \frac{1}{2} kT_e \\
\Rightarrow V_e = \sqrt{3kT_e/m_e} \\
\text{These relations then give} \\
\lambda_e = \frac{\sqrt{3}}{16\pi} \frac{k_\beta(T_e)^{3/2}}{m_e^{5/2} Z_e^4 \ln(Z_e)} \\
\text{Note } \lambda_e \sim T_e^{5/2} \\
\text{predictor is better with more sophisticated theory} \\
\text{too small in our simple theory} \\
\text{Spitzer-Harm Theory: Prefactor } = 40 \sqrt{2\pi} \approx 100 \text{ for } Z = \infty \\
\text{much larger } \rightarrow e^- \text{ transport is predominantly from hot } e^- \text{ in tail of Maxwellian} \\
\text{Better still } \text{Degrassini: our predictor too small by } \times 3 \text{ for } Z = 1 \text{ or } Z = \infty \\
\text{Example: Short pulse laser heating} \\
kT_e = 100 \text{ eV} \\
\frac{2}{\lambda_e} = 5 \text{ cm}^{-1} \text{ cm}^{-3} \\
\lambda_e = 10^{-5} \text{ cm}^{-1} \\
Q \approx \frac{1}{A} \frac{dE}{dx} \Rightarrow \tau_e \approx \frac{4 \times 10^{-3}}{\lambda_e} \\
\delta t \approx 4 \text{ ps}
III. B  Electron heat conduction in large gradients

Thermal gradient: \( \delta x \sim \frac{T_e}{\sqrt{v_e}} \)

What is \( \delta x \) becomes small
\( \Rightarrow S \) becomes huge, since \( S \sim \frac{T_e}{\delta x} \)

But: \( \frac{e^-}{e^+} \rightarrow \frac{e^-}{e^+} \text{ plane} \Rightarrow \frac{S}{e^-} \) cannot be larger than the rate at which free streaming electrons fly across this plane.

So there is a **maximum rate of heat flux**

\[ S_{\text{max}} = n_e v_e E_e \]
\( E_e \) = energy

\[ S_{\text{max}} = \left( \frac{3}{2} \right) n_e \sqrt{k_e T_e} \]

So we can define **flux limited electron heat flow**

\[ S = -S_e \sqrt{T_e} \text{ for } S < f S_{\text{max}} \]
\[ = -f S_{\text{max}} \text{ otherwise} \]

where \( f \) is the flux limiter → found empirically

\( f \sim 0.05 - 0.1 \) usually

More sophisticated models include nonlocal effects

\[ k_e T_e \]

Heat flux at this point is a function of electrons flowing from all points within one MFP, \( R_e \).
A phenomenological model is to say

\[ S(x) = \frac{1}{B} \int_{-\infty}^{\infty} S_{\text{SH}}(x') G(x, x') \, dx' \]

where \( S_{\text{SH}} = X_0 \Delta T_0 \)

A common non-local model is to say

\[ G(x, x') = \frac{1}{2a \lambda_c} \exp \left[ -\frac{1}{a} \frac{|x-x'|}{\lambda_c} \right] \]

where \( a \) is a free parameter \( a \approx 30 \)

From numerical simulations

\[ (lx) \]

\( \gamma \) is flux here is integral of flux from a distance \( a \lambda_c \) away

\[ S_{\text{SH}} \]

III. C. Radiative Heat Conduction

Consider energy carried by photons of one frequency \( \nu \)

\( U_\nu \) is energy density of photons of freq. \( \nu \).

\( \lambda_\nu \) = photon mean free path

By arguments similar to the above

\[ S_{\lambda} = -\frac{1}{3} c \lambda \nu \frac{dU_\nu}{d\lambda} \]
But this is for only one photon energy
and \( \nu \) will have a complicated dependence
on \( \nu \) because of atomic effects.
So now consider the total heat flux

\[
\overline{S}_{tot} = \int_0^\infty \overline{S}_\nu \ d\nu = -\frac{C}{3} \int_0^\infty \nu \ \nabla U_\nu \ d\nu
\]

If the plasma is optically thick the energy
density of photons is given by a
Planckian:

\[
U_\nu = \frac{8\pi h\nu^3}{c^2} \frac{1}{\exp [h\nu/kT] - 1}
\]

With total radiation density

\[
U_{tot} = \int U_\nu \ d\nu = 4 \sigma_0 T^4 / c
\]

\( \sigma_0 \) = Stefan-Boltzmann constant.

We would like a simple diffusive eq. for
the radiation heat flux:

\[
\overline{S} = -\frac{\lambda R c}{3} \nabla U_{tot} \quad \lambda R \text{ is some appropriately averaged photon mass}
\]

\[
\overline{S} = -\frac{16 \sigma_0 \nu R T^3}{3} \nabla T
\]

\( \nu R \) = radiation heat conductivity

So we need to find \( \lambda R \)

\[
\overline{S} = -\frac{\lambda R c}{3} \nabla U_{tot} = -\int \frac{c}{3} \nu \ \nabla U_\nu \ d\nu
\]

\[
\frac{dU}{dx} = \frac{dU}{dT} \frac{dT}{dx} = \frac{dU}{dT} \nabla T
\]
\[-\frac{\hbar^2}{3} \frac{\partial U_{\nu}}{\partial T} \Delta T = -\frac{c}{3} \int_{0}^{\infty} \nu \frac{\partial U_{\nu}}{\partial \nu} \Delta T \, d\nu\]

\[
\frac{dU_{\nu}}{dT} = 16 \nu_0 \ T^3/c = \frac{32 \pi^5 \ h^5 \ T^3}{15 \ h^3 \ c^3}
\]

\[
\frac{dU_{\nu}}{dT} = \frac{8 \pi^5 \ h^2 \ T^2}{c^3 \ h_0 \ T^2} \left( \frac{e^{\hbar \nu/kT}}{e^{\hbar \nu/kT} - 1} \right)^2
\]

So,

\[
\lambda_R = \frac{15 \ h^3 \ c^3}{32 \pi^5 \ h_0 \ T^3} \int \nu \frac{8 \pi^5 \ h^2 \ T^2}{c^3 \ h_0 \ T^2} \left( \frac{e^{\hbar \nu/kT}}{e^{\hbar \nu/kT} - 1} \right)^2 \, d\nu
\]

One more consideration,

\[
\lambda_{\nu} = 1/\lambda_{\nu} = \text{absorption length of photons}
\]

\[
\text{determined by complicated quantum level structure}
\]

\[
\text{this level is also populated}
\]

So, not only can there be absorption, it can be slightly offset by stimulated emission.

So effective absorption is \( \lambda_{\nu} = \lambda_{\nu} \cdot (1 - e^{-\hbar \nu/kT}) \)

We have to divide \( \lambda_{\nu} \) by an additional factor of \( \frac{1}{(1 - e^{-\hbar \nu/kT})} \)

Then,

\[
\lambda_R = \int_{0}^{\infty} \frac{15 \ h^3}{4 \pi^5} \left( \frac{h \nu}{kT} \right)^4 \frac{1}{kT} \left( \frac{e^{\hbar \nu/kT}}{e^{\hbar \nu/kT} - 1} \right)^2 \lambda_{\nu} \, d\nu
\]

\( \lambda_R \) is known as the Rosseland Mean Free Path.

It is a weighting of the photon mean free path properties of the plasma.
Absorption mechanisms for photons in a plasma

1) Free - Free → Inverse Bremsstrahlung
2) Bound - Free → photo ionization
3) Bound - Bound → line absorption

In low Z fully ionized (1) dominates

\[ \lambda_{\nu} \propto \frac{4}{3} \left( \frac{2 \pi}{3 m_e k T} \right)^{\frac{2}{3}} \frac{Z^2 e^6}{\hbar^2 \nu^2} \]

which leads to \( \lambda_{\nu} \propto (kT)^{\frac{5}{6}} \)

\( \lambda \propto T^{\frac{5}{6}} \) very nonlinear

Much more complex with partially ionized ions

Bound - Free and Bound - Bound dominate

\[ \lambda_{\nu} \propto 3 \frac{b^4 e^3}{2^7 e^b n_i T} (kT)^3 \]

\[ \frac{\sigma_{ii}}{3} \frac{16}{5} \lambda_{\nu} T^3 \Rightarrow \lambda \propto T^6 \]
Note that \( \frac{\chi_{\text{rad}}}{\chi_{\text{nh}}} \approx 5 \times 10^{18} \frac{(kT_e)^{5/2}}{n_e} \)

\( kT_e \) in eV \( n_e \) in cm\(^{-3}\)

So at solid density \( \chi_{\text{rad}} \approx \chi_{\text{nh}} \)

\( n_e = 10^{23} \) \( kT_e \geq (5 \times 10^{-5})^{-2/5} \)

\( kT_e \geq 50 \) eV

\[ \Rightarrow \]

However, line absorption becomes \( \lambda^2 \) at 50 eV

...dominates to higher \( kT_e \)

**III. D Thermal Waves**

Consider:

\[ \begin{align*}
\text{Heating} & \quad \text{radiation} \\
\text{\( \Delta T \)} & \quad \text{\( \frac{1}{5} \)} \quad \text{\( \rightarrow \)}
\end{align*} \]

- If energy deposition is fast (ie short pulse laser)

\[ \begin{align*}
T_e & \quad t=0 \\
\Delta T_e & \quad t=\text{max} \\
T_e & \quad t=2
\end{align*} \]

\[ \begin{align*}
X_e(0) & \quad X_e(1) \quad X_e(2)
\end{align*} \]

Note: non-linearity in \( \chi(T) \) leads to a steepening of the thermal front.

\[ \begin{align*}
T & \quad \text{sharp front}
\end{align*} \]

\[ \begin{align*}
S & \quad \text{non-linear with } T
\end{align*} \]

\[ \begin{align*}
\frac{dS}{dx} & \quad \text{cooling}
\end{align*} \]

\[ \begin{align*}
\frac{dS}{dx} & \quad \text{heating}
\end{align*} \]

\[ \frac{1}{X_e} \]

\[ \frac{1}{X_e} \]
Let's consider a thermal wave with

\[ X = u \cdot T^n \quad n = \frac{5}{2} \text{ for } c^- \text{ conduction} \]

\[ n = 6 - 6 \frac{x}{x} \text{ for radiation conduction} \]

Wave is self-similar

\[ S = x \cdot \frac{2T}{dx} = u \cdot T^n \cdot \frac{dT}{dx} \]

\[ \frac{dT}{dt} = u \cdot \frac{2T}{dx} \cdot \frac{dT}{dx} \]

We shall say that \( E \) erg/cm\(^2\) was deposited on the plasma surface.

\[ n \cdot k_\beta = E \quad U \text{ in units deg cm} \]

Since self-similar we can find a dimensionless quantity such that

\[ X \rightarrow x \quad t \rightarrow a \cdot t \]

only \( u \) and \( U \) enter the problem.

Units of \( u \) are cm\(^2\) deg\(^n\).

Units of \( U \) are deg cm.

So the only dimensionless quantity that contains \( a, U, x, t \) is

\[ \alpha = \frac{x}{(a \cdot U^n \cdot t) \text{ Unit}} \]

So this implies a thermal wave front \( X_p \)

\[ \alpha_0 = \frac{x_p}{(a \cdot U^n \cdot t) \text{ Unit}} \implies X_p(t) = \alpha_0 \cdot \sqrt{U \cdot \text{Horiz}} \cdot t \cdot \text{Amp}^2 \]

of order unity.
So the trajectory of the thermal wave front

\[ x_e \sim t^{1/3} \]

\[ x_e \sim t^{2/3} \quad \text{for an electron driven wave} \]

\[ x_e \sim t^{1/2} \quad \text{for a radiation driven wave} \]

very fast at first

\[ \Rightarrow \text{Case 2} - \text{constant incident heat flux} \]

\[ N \text{ov. Tem.} \]

If we have energy/cm²-s incident by conservation of energy

\[ \varepsilon_0 \cdot t / x_e \approx \frac{3}{2} kT \]

From heat flux eq.

\[ \varepsilon_0 = a T^n \frac{2T}{x} \approx a \frac{T^{n+1}}{x_e} \]

Eliminating \( T \)

\[ x_e \approx \varepsilon_0^{1/(n+2)} a^{1/2} t^{1/(n+2)} \approx t^{2/3} \quad \text{for electron} \]

Almost linear

\[ \approx t^{7/6} \quad \text{for radiation} \]