LECTURE # 2

One dimensional implosion hydrodynamics

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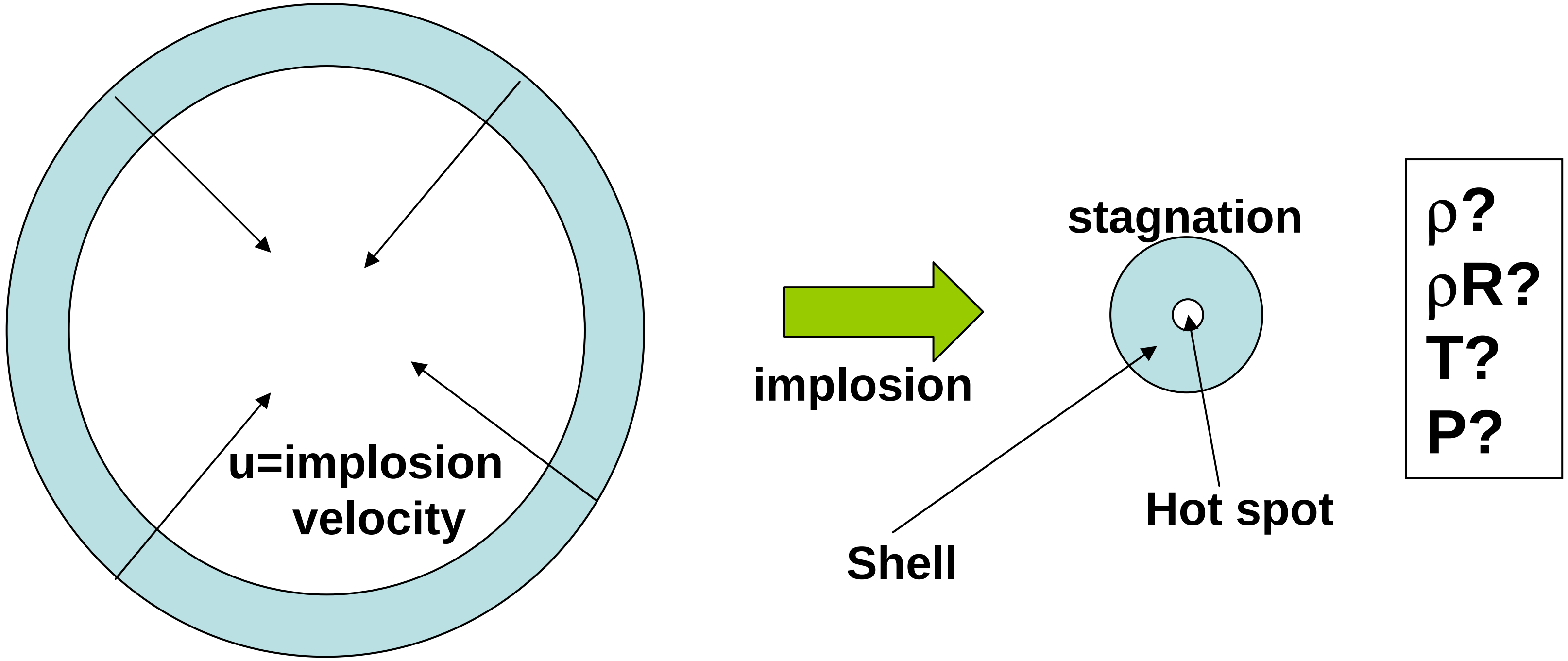
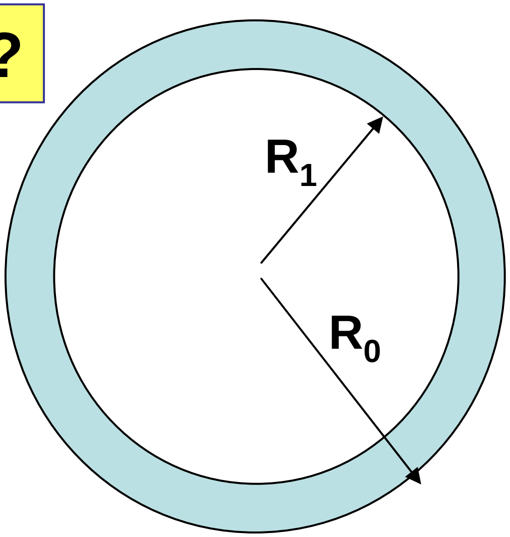
1D implosion hydrodynamic theory should answer the following question:

- **What are the stagnation values of the relevant hydrodynamic properties?**



What variables can be controlled?



(a) Shell inner radius R_0 at time t=0

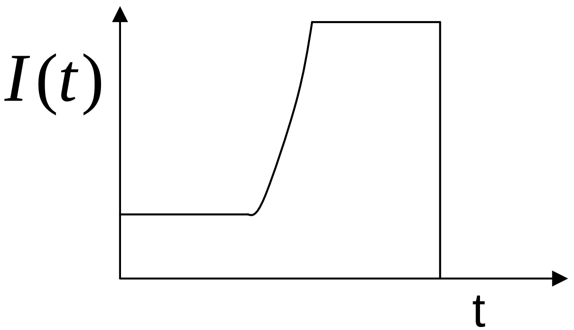
(b) Shell outer radius R_1 at time t=0

(c) The total laser energy on target

(d) Adiabat (or Entropy) through shocks

(e) Applied pressure P(t) through the pulse shape I(t) ←laser intensity

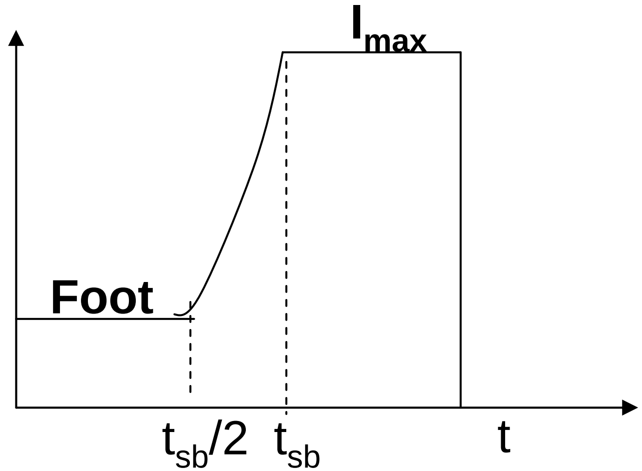
$$\alpha \sim \frac{P}{\rho^{5/3}} \qquad P \sim I^{2/3}$$



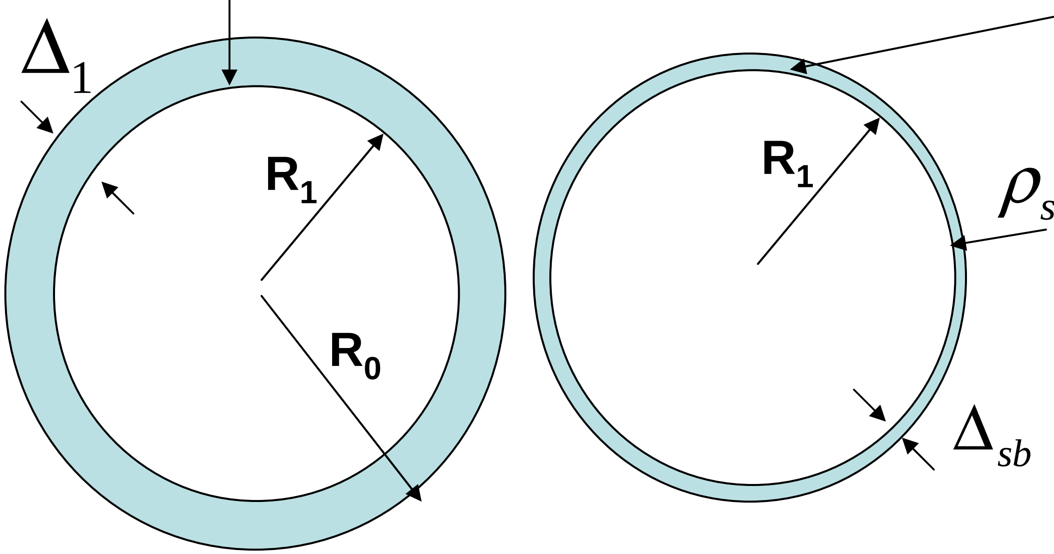
The adiabat is set by the foot of the laser pulse

$$\alpha \sim \frac{P_{foot}}{(4\rho_1)^{5/3}}$$

$$I(t)$$



$$\rho_1 = initial\ density$$



$$\rho_{sb} \sim \left(\frac{P_{\max}}{\alpha}\right)^{3/5} = 4\rho_1\left(\frac{P_{\max}}{P_{foot}}\right)^{3/5}$$

$$\Delta_{sb} \sim \frac{\Delta_1}{4}\left(\frac{P_{foot}}{P_{\max}}\right)^{3/5} \sim \frac{\Delta_1}{4}\left(\frac{I_{foot}}{I_{\max}}\right)^{2/5}$$

Density and thickness at shock break out

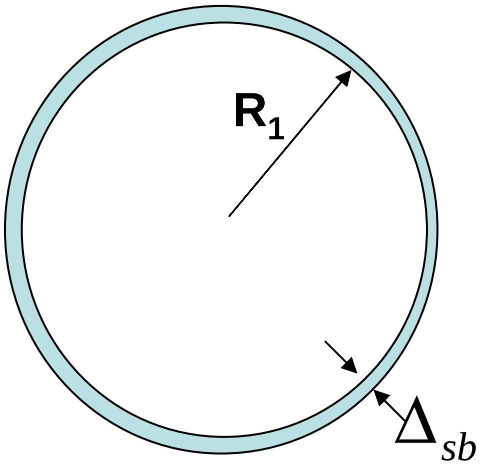
- **Use** $p \sim I^{2/3}$ **to find:**

Density→ $$\rho_{sb} \sim 4\rho_1 \left(\frac{I_{\max}}{I_{foot}} \right)^{2/5}$$

Shell thickness → $$\Delta_{sb} \sim \frac{\Delta_1}{4} \left(\frac{I_{foot}}{I_{\max}} \right)^{2/5}$$

Shell radius → $$\boldsymbol{R} \approx \boldsymbol{R}_1$$

At shock break out the aspect ratio is maximum



$$A_1 = \frac{R_1}{\Delta_1} = \text{initial aspect ratio}$$

$$A_{sb} = IFAR = \frac{R_1}{\Delta_{sb}} = 4A_1\left(\frac{I_{\max}}{I_{foot}}\right)^{2/5}$$

IFAR = Aspect ratio at shock break-out = Maximum In-Flight-Aspect-Ratio

$$A_{sb} = A_{\max}$$

The IFAR scales with the Mach number

The shell kinetic energy is equal to the work done on the shell (this relation should be improved by including the ablated mass)

$$Mu_{\max}^2 \sim \int_R^{R_1} pr^2 dr \sim p(R_1^3 - R^3)$$

Neglect for R<<R_0

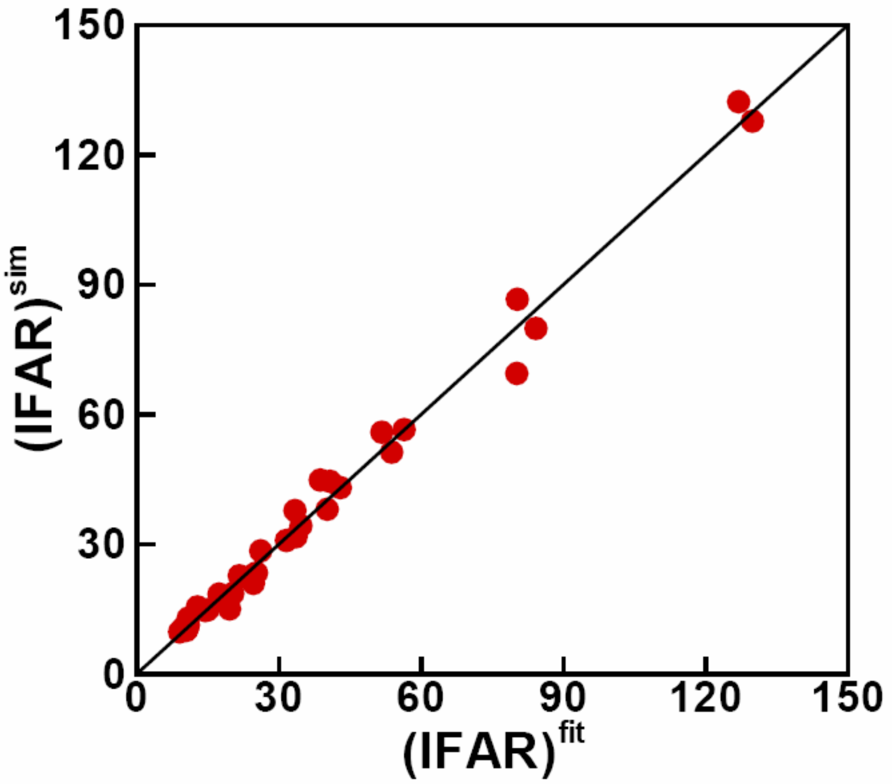
$$M \sim \rho_{sb} \Delta_{sb} R_1^2$$

$$IFAR = A_{sb} = \frac{R_1}{\Delta_{sb}} \sim \frac{u_{\max}^2}{p/\rho_{sb}} \sim Mach_{\max}^2$$

$$\rho \sim (p/\alpha)^{3/5} \qquad p \sim I^{2/3}$$

$$IFAR = A_{sb} = \frac{R_1}{\Delta_{sb}} \sim \frac{u_{\max}^2}{\alpha^{3/5} I^{4/15}}$$

1D simulations confirm the IFAR scaling with the Mach number



$$IFAR_{\max}^{fit} = \frac{55 I_{15}^{-0.27}}{\alpha^{0.72}} \left(\frac{u_{\max}(cm/s)}{3 \bullet 10^{7}} \right)^{2.12}$$

The IFAR formula can be used to find the final implosion velocity.

$$u_{\max}^2 \sim IFAR \bullet \alpha^{3/5} I^{4/15}$$

Substituting the IFAR from shock breakout, we find the implosion velocity as a function of control variables

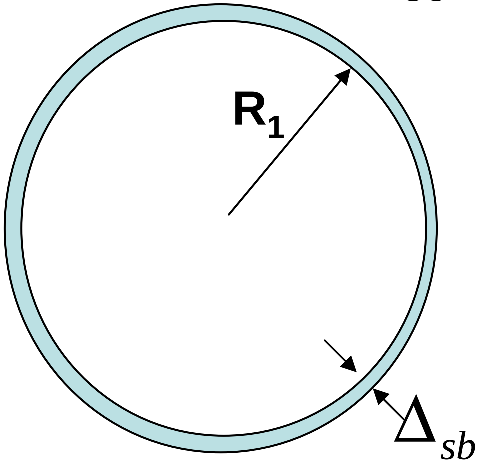
$$IFAR = 4A_1 \left(\frac{I_{\max}}{I_{foot}} \right)^{2/5}$$

$$u_{\max}^{cm/s} \approx 10^7 \sqrt{0.7 A_1 \bullet \alpha^{3/5} I_{15(\max)}^{4/15} \left(\frac{I_{\max}}{I_{foot}} \right)^{2/5}}$$

A simple implosion theory can be derived in the limit of infinite aspect ratio at shock break-out

- Consider a high aspect ratio shell at the beginning of the acceleration phase (maximum aspect ratio)

$$IFAR = A_{sb} = \frac{R_1}{\Delta_{sb}} >> 1$$



A first step in implosion hydrodynamic theory is the paper of Basko and Meyer-ter-vehn, Phys. Rev. Lett. 88, 244502-1 (2002)

There are two important time scales:
the shell relaxation time and the implosion time

Shell relaxation: $t_{rel} \sim \Delta / C_s$

Implosion time: $t_i \sim R / u_i$

$$\frac{t_i}{t_{rel}} \sim \frac{A}{Mach} \qquad A = \frac{R}{\Delta} \qquad Mach = \frac{u_i}{C_s}$$

In the acceleration phase $A \sim Mach^2$

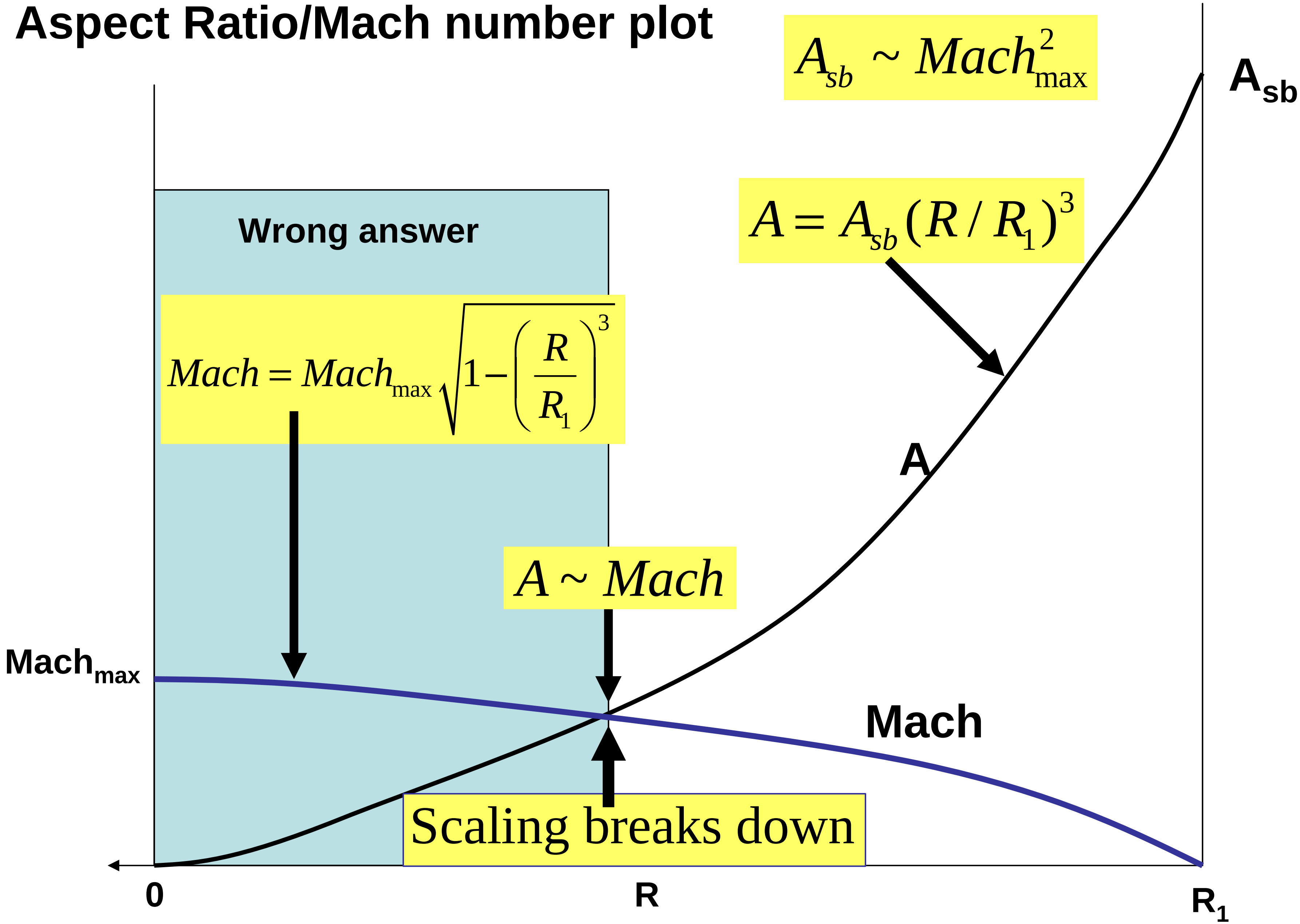
The density is constant

$$\frac{t_i}{t_{rel}} \sim \frac{A}{Mach} \sim Mach \sim \sqrt{A} >> 1 \rightarrow \rho \approx const$$

$$\Delta \sim 1 / R^2$$

$$A = A_{sb} (R / R_1)^3$$

Aspect Ratio/Mach number plot



The acceleration phase ends when the implosion time is of the order of the relaxation time at $R_* \sim R_1 / A_{sb}^{1/6}$

Start from $$A = A_{sb}\left(\frac{R}{R_1}\right)^3$$ **with** $A_{sb} \sim Mach_{\max}^2$ **and** $R << R_1$

The relaxation time is of the same order of → the implosion time

$$\frac{t_i}{t_{rel}} \sim \frac{A}{Mach_{\max}} \sim 1$$ **when** $A \sim Mach_{\max}$

$$A \sim A_* \sim A_{sb}\left(\frac{R_*}{R_1}\right)^3 \sim Mach_{\max} \sim A_{sb}^{1/2}$$

When $$R_* / R_1 \sim 1/A_{sb}^{1/6} \sim 1/Mach_{\max}^{1/3}$$

For $R/R_1<1/A_{sb}^{1/6}$ the coasting phase begins and the shell thickness Δ is constant

For $R<<R_1$ the force applied to the shell is small since $4\pi pR^2$ is small → the shell keeps imploding at constant velocity → coasting phase

$$(R/R_1)<(R_*/R_1)\sim 1/A_{sb}^{\;1/6}$$

The implosion time left (R_*/u_{max}) is less than the relaxation time ⇒

$$\frac{t_i}{t_{rel}}<1$$

The shell thickness is constant ⇒

$$\Delta\sim const=\Delta_*$$

$$\Delta_*\sim\frac{\Delta_*}{R_*}R_*\sim\frac{R_*}{A_*}\sim\frac{R_*}{R_1}\frac{R_1}{A_*}\sim\frac{1}{A_{sb}^{1/6}}\frac{R_1}{A_{sb}^{1/2}}\sim\frac{R_1}{A_{sb}^{2/3}}$$

In the coasting phase the Mach number decreases like $R^{2/3}$ and the aspect ratio like R

$$A = \frac{R}{\Delta_*} = \frac{R_*}{\Delta_*}\frac{R}{R_*} = A_*\frac{R}{R_*}$$

$$Mass \sim \rho R^2 \Delta_* \Rightarrow \rho \sim 1/R^2$$

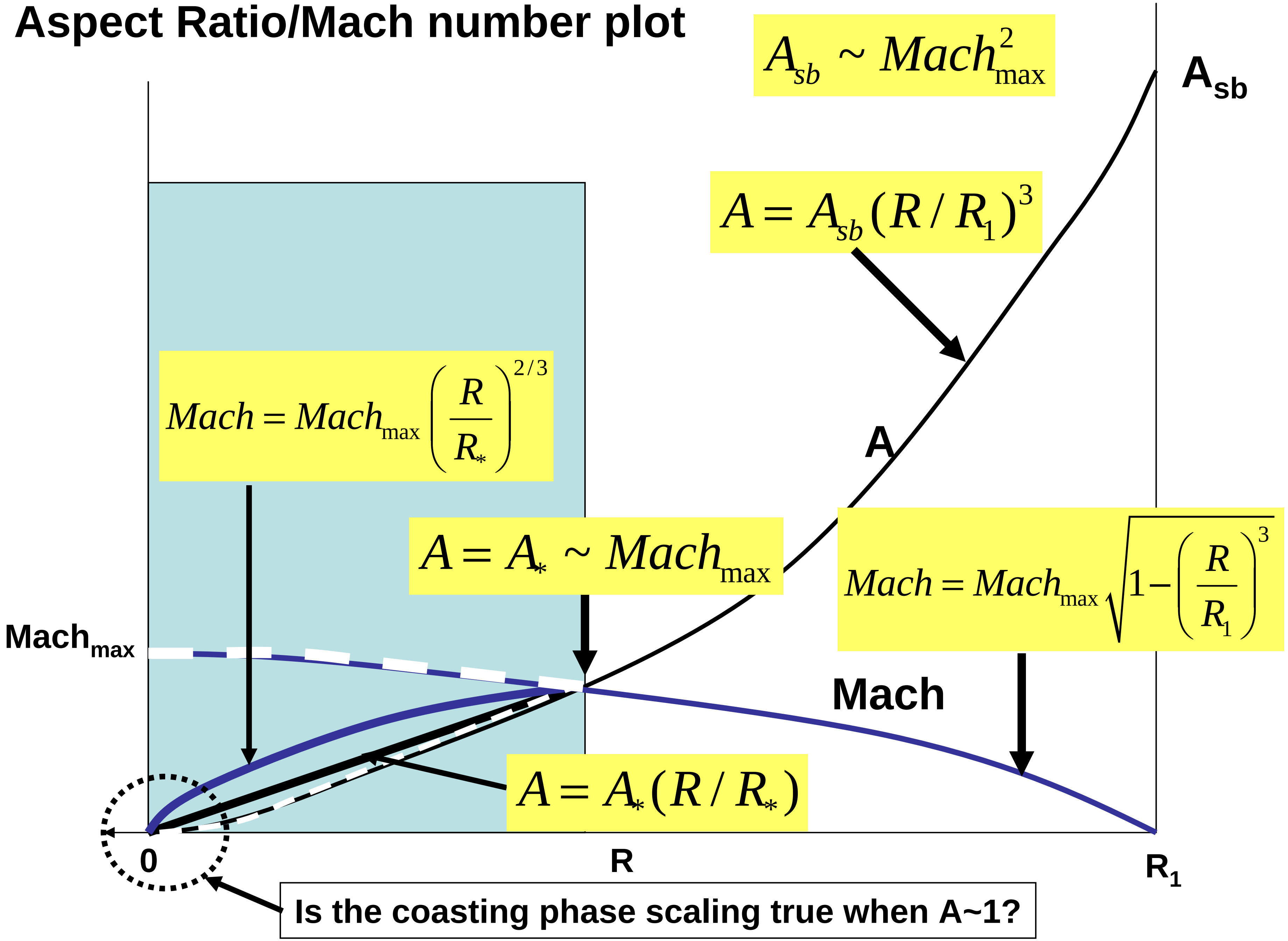
$$Mach \sim \frac{u_{\max}}{\sqrt{p/\rho}}$$

$$p \sim \alpha \rho^{5/3}$$

$$Mach \sim R^{2/3} \Rightarrow Mach = Mach_{\max}(R/R_*)^{2/3}$$

$$Mach \sim \sqrt{A_{sb}}(R/R_*)^{2/3}$$

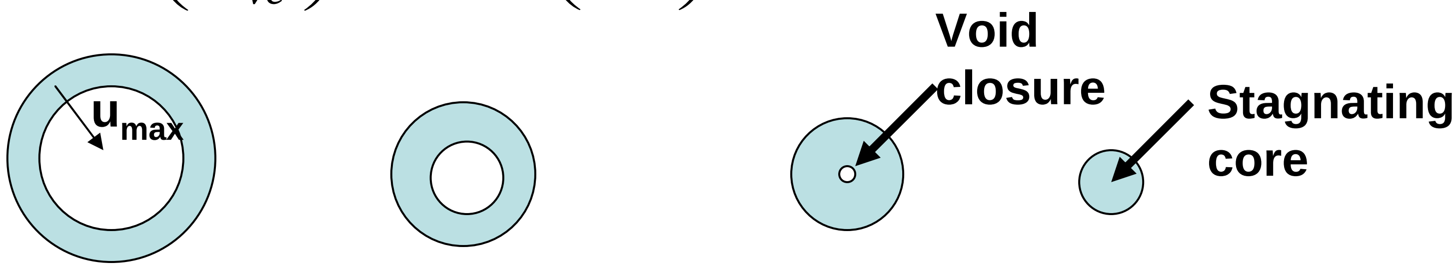
Aspect Ratio/Mach number plot



A scaling law can be derived for the density at stagnation

Before "void closure" (vc) the Aspect Ratio A~1 → $R_{vc} \sim \Delta_*$

$$\rho_{vc} = \rho_* \left(\frac{R_*}{R_{vc}}\right)^2 = \rho_{sb}\left(\frac{R_*}{\Delta_*}\right)^2 = \rho_{sb} A_*^2 = \rho_{sb} Mach_{\max}^2$$



The collapse of the shell generates a "return' shock that propagates from the center outward. The return shock compresses the core by a factor of 4 (not quite right)

$$\rho_{stag} \sim 4\rho_{vc} \sim 4\rho_{sb} Mach_{\max}^2 \sim 4\rho_{sb} A_{sb}$$

Use $\dfrac{\rho_{sb}}{\rho_1} \sim 4\left(\dfrac{I_{\max}}{I_{foot}}\right)^{2/5}$ ⇒ $\rho_{stag} \sim 16\rho_1 IFAR\left(\dfrac{I_{\max}}{I_{foot}}\right)^{2/5}$

The stagnation pressure scaling follows from energy conservation. The stagnation entropy scales as Mach$^{2/3}$

Energy → conservation $P_{stag} R_{stag}^3 \sim Mass \times u_{\max}^2$ $\qquad Mass \sim \rho_{stag} R_{stag}^3$

$$P_{stag} \sim \rho_{stag} \times u_{\max}^2 \sim P_{applied} \frac{\rho_{stag}}{\rho_{sb}} \frac{u_{\max}^2}{P_{applied} / \rho_{sb}}$$

Stagnation pressure scaling

$$P_{stag} \sim P_{applied} Mach_{\max}^4 \sim P_{applied} IFAR^2$$

Stagnation entropy scaling

$$\alpha_{stag} \sim \frac{P_{stag}}{\rho_{stag}^{5/3}} \sim \alpha \, Mach_{\max}^{2/3} \sim \alpha \, IFAR^{1/3}$$

Scaling of the areal density of the compressed core

$$M = \frac{Mu^2}{u^2} \sim \frac{E}{u^2}$$

$$\rho_{stag} R_{stag} \sim \rho_{stag}^{2/3} (\rho_{stag} R_{stag}^3)^{1/3} \sim \left(\rho_{sb} Mach_{\max}^2\right)^{2/3} \left(\frac{E}{u^2}\right)^{1/3}$$

$$\rho_{sb} \sim \left(\frac{P_{applied}}{\alpha}\right)^{3/5}$$

$$P_{applied} \sim I^{2/3}$$

$$Mach_{\max}^2 \sim \frac{u_{\max}^2}{\alpha^{3/5} I^{4/15}}$$

The areal density ρR scales as the laser energy E to the power 1/3.

$$\rho_{stag} R_{stag} \sim \frac{E^{1/3} u_{\max}^{2/3} I^{4/45}}{\alpha^{4/5}}$$

Here I consider a fixed laser wavelength

Amplification of the areal density

$$\rho_{stag} R_{stag} \sim \rho_{stag}^{2/3} (\rho_{stag} R_{stag}^{3})^{1/3} \sim \rho_{sb}^{2/3} Mach_{\max}^{4/3} M^{1/3}$$

$$\rho_{stag} R_{stag} \sim \frac{\rho_{sb}^{2/3}}{\rho_{1}^{2/3}} Mach_{\max}^{4/3} \rho_{1}^{2/3} (\rho_{1} R_{1}^{2} \Delta_{1})^{1/3}$$

$$\rho_1 A_1^2 \Delta_1^3$$

$$\rho_{stag} R_{stag} \sim \left(\rho_1 \Delta_1\right) Mach_{\max}^{4/3} A_1^{2/3} \left(\frac{\rho_{sb}}{\rho_1}\right)^{2/3}$$

where $$\frac{\rho_{sb}}{\rho_1} = 4\left(\frac{I_{\max}}{I_{foot}}\right)^{2/5}$$

IMPLOSION SCALING RECAP

$$A_{sb} = IFAR = 4A_1\left(\frac{I_{\max}}{I_{foot}}\right)^{2/5}$$

$$u_{\max}^{cm/s} \approx 10^7\sqrt{0.7A_1 \bullet \alpha^{3/5} I_{15(\max)}^{4/15}\left(\frac{I_{\max}}{I_{foot}}\right)^{2/5}}$$

$$\rho_{stag} \sim \rho_{sb} Mach_{\max}^2 \sim 16\rho_1 IFAR\left(\frac{I_{\max}}{I_{foot}}\right)^{2/5}$$

$$P_{stag} \sim P_{applied} Mach_{\max}^4 \sim P_{applied} IFAR^2$$

$$\alpha_{stag} \sim \alpha Mach^{2/3} \sim \alpha IFAR^{1/3}$$

$$(\rho R)_{stag} \sim 4^{2/3}(\rho_1\Delta_1)IFAR^{2/3}A_1^{2/3}\left(\frac{I_{\max}}{I_{foot}}\right)^{4/15}$$

And the laser energy?

- The laser energy is included through the peak laser intensity and the shell outer radius

$$E_L = 4\pi R_0^2 I_{\max} t_{imp} \approx 4\pi R_0^2 I_{\max} \frac{R_0}{u_{\max}}$$

$$E_L \approx \frac{4\pi R_0^3 I_{\max}}{u_{\max}}$$

Relation between laser energy and implosion velocity
(does not include effect of ablated mass)

$$u_{\max}^{cm/s} \approx 10^7 \sqrt{0.7 A_1 \bullet \alpha^{3/5} I_{15(\max)}^{4/15} \left(\frac{I_{\max}}{I_{foot}} \right)^{2/5}}$$

$$\alpha = \frac{P_{foot}(Mbar)}{2.2(4\rho_1^{g/cc})^{5/3}} \approx \frac{P_{foot}(Mbar)}{2.2} \approx \frac{83\left(I_{15}^{foot}\right)^{2/3}}{2.2}$$

$$u_{\max}^{cm/s} \approx 2.5 \bullet 10^7 \sqrt{A_1 I_{15(\max)}^{2/3}}$$

$$\boldsymbol{E_L \approx \frac{4\pi R_0^3 I_{\max}}{u_{\max}}}$$

$$u_{\max}^{cm/s} \approx 10^7 A_1^{1/4} \sqrt{\frac{E_L(kJ)}{100} \frac{1}{\Delta_{1(mm)} R_{0(mm)}^2}}$$

Homework problem

- **Consider typical target shown in previous lecture**
- **$A_1 \approx 4$, $R_1 = 1350\ \mu m$, $R_0 = 1700 \mu m$, $\Delta_1 = 350\ \mu m$**
- **$\rho_1 = 0.25$ g/cc**
- **$\rho_1 \Delta_1 = 0.009$ g/cm^2**
- **$I_{max} = 10^{15}$ W/cm^2 leading to $P_{max} \sim 100$Mbar**
- **$I_{foot} = 2.2\ 10^{13}$ W/cm^2 to set the shell on $\alpha = 3$**

Use hydro-relations in previous slide to find:

$$IFAR \approx 70 \qquad u_{max} \approx 5 \bullet 10^7\, cm/s$$

$$\rho_{stag} \approx 1000\, g/cc \quad \Rightarrow \quad \textbf{amplification} \sim 4000$$

$$(\rho R)_{stag} \approx 2\, g/cm^2 \quad \Rightarrow \quad \textbf{amplification} \sim 200$$

$$P_{stag} \approx 500\, Gbar \quad \Rightarrow \quad \textbf{amplification} \sim 5000$$

$$E_{Laser} \sim 1.3\, MJ$$

Hydrodynamic Scaling Relations from Simulations

Variable	Scaling Relation
Hydrodynamic Efficiency	$\eta \approx \frac{0.051}{I_{15}^{0.25}}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{0.75}\left(\frac{0.35}{\lambda_L(\mu m)}\right)^{0.5}$
Thermonuclear Gain	$G \approx \frac{365}{I_{15}^{0.25}}\left(\frac{3\times10^{7}}{u_{\max}(cm/s)}\right)^{1.25}\left(\frac{\rho R(g/cm^2)}{7+\rho R}\right)\left(\frac{0.35}{\lambda_L(\mu m)}\right)^{0.5}$
Ignition Energy (MJ)	$E_L^{ign} \approx 0.64 I_{15}^{-0.26}\alpha_{inn}^{1.9}\left(\frac{3\times10^{7}(cm/s)}{u_{\max}}\right)^{6.6}\left(\frac{\lambda_L(\mu m)}{0.35}\right)$
Shell Areal Density (g/cm²)	$(\rho R)_{\max} \approx \frac{1.2}{\alpha_{inn}^{0.54}}\left(\frac{E_L(kJ)}{100}\right)^{0.33}\left(\frac{0.35}{\lambda_L(\mu m)}\right)^{0.25}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{0.06}$
Shell Density (g/cm³)	$\langle\rho\rangle_{\rho R} \approx \frac{425}{\alpha_{inn}^{1.12}} I_{15}^{0.13}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)\left(\frac{0.35}{\lambda_L(\mu m)}\right)^{0.13}$
Shell IFAR	$IFAR \approx \frac{40 I_{15}^{-0.27}}{\langle\alpha_{IF}\rangle^{0.72}}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{2.12}\left(\frac{\lambda_L(\mu m)}{0.35}\right)^{0.27}$
Hot spot Areal Density (g/cm²)	$\rho R_h \approx \frac{0.31}{\alpha_{inn}^{0.55}}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{0.62}\left(\frac{E_L(kJ)}{100}\right)^{0.27}$
Hot spot Temperature (keV)	$\langle T_h\rangle \approx \frac{2.96}{\alpha_{inn}^{0.15}}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{1.25}\left(\frac{E_L(kJ)}{100}\right)^{0.07}$
Hot spot Pressure (Gbar)	$\langle p_h\rangle \approx \frac{345}{\alpha_{inn}^{0.90}}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{1.85}$
Stagnation Aspect Ratio	$A_s \approx \frac{1.48}{\alpha_{inn}^{0.19}}\left(\frac{u_{\max}(cm/s)}{3\times10^{7}}\right)^{0.96}$

*** Ref. C.D. Zhou and R. Betti PoP paper. With the exception of the gain, all the variables are calculated in the absence of alpha-particle deposition. The gain is calculated by assuming that ignition has taken place.**